# More Useful Antiderivative formulas (memorize)

| *Function* | *Antiderivative* |
|---|---|
| $\boxed{e^{ax}}$ | $\boxed{\frac{1}{a}e^{ax}}$ |
| $\frac{1}{x}$ | $\ln\lvert x\rvert$ |
| $\frac{1}{\sqrt{1-(ax)^2}}$ | $\frac{1}{a}\sin^{-1}(ax)$ |
| $\frac{1}{1+(ax)^2}$ | $\frac{1}{a}\tan^{-1}(ax)$ |
| $b^{ax}$ | $\frac{1}{a\ln b}b^{ax}, b>0, b\neq 1$ |

In the previous example, $a=-2$

$\rightarrow \int e^{-2x}\,dx = -\frac{1}{2}e^{-2x}+C$

(know these)

We will cover this one on Wednesday! :-)

(Understand how to quickly derive this formula)

$x = e^{\ln(x)}$

Preview: $b^{ax} = e^{ax\cdot\ln(b)} \rightarrow \frac{d}{dx}\left[b^{ax}\right] = a\ln(b)\cdot b^{ax}$

# Example 3.1:

Evaluate the following indefinite integral:

$$\int (2\tan x + 1)\sec^2(x)dx = \int 2\tan x \sec^2 x\, dx * + \int \sec^2 x\, dx *$$

$f' = 2x \rightarrow f(x) = x^2$

$\int \sec^2 x\, dx = \tan x + C$

$\frac{d}{dx}[\sec x] = \tan x \sec x$

$\int 2\underbrace{\tan x \cdot \sec x}\cdot \underbrace{\sec x}\, dx$

$= \sec^2 x + C$

$$\frac{d}{dx}\left[\sec^2 x\right] = 2\sec x \cdot \frac{d}{dx}\left[\sec x\right]$$

$$= \tan x + \sec^2 x + C$$

# Example 3.2:

recall: $\int \frac{dx}{\sqrt{1-(ax)^2}} = \frac{1}{a}\sin^{-1}(ax) + C$

Evaluate the following indefinite integral:

$$\int \frac{dx}{\sqrt{16-x^2}}$$

$\sqrt{16-x^2} = \sqrt{16}\sqrt{1-\left(\frac{x}{4}\right)^2}$

$= \frac{1}{4}\int \frac{dx}{\sqrt{1-(x/4)^2}}$ $(a = 1/4)$

$= \frac{1}{4} \cdot 4 \sin^{-1}\left(\frac{x}{4}\right) + C = \sin^{-1}\left(\frac{x}{4}\right) + C$

# Example 3.3:

How would you find a formula for the following indefinite integral?

$$\int \frac{dx}{x^2 - x + 1}$$

brainstorm:

→ complete the square

→ make it look like a $\tan^{-1}(\ldots)$ antiderivative

# Section 5.1-5.3: Area under the curve and the definite integral

Math 1552 lecture slides adapted from the course materials
By Klara Grodzinsky (GA Tech, *School of Mathematics*, Summer 2021)

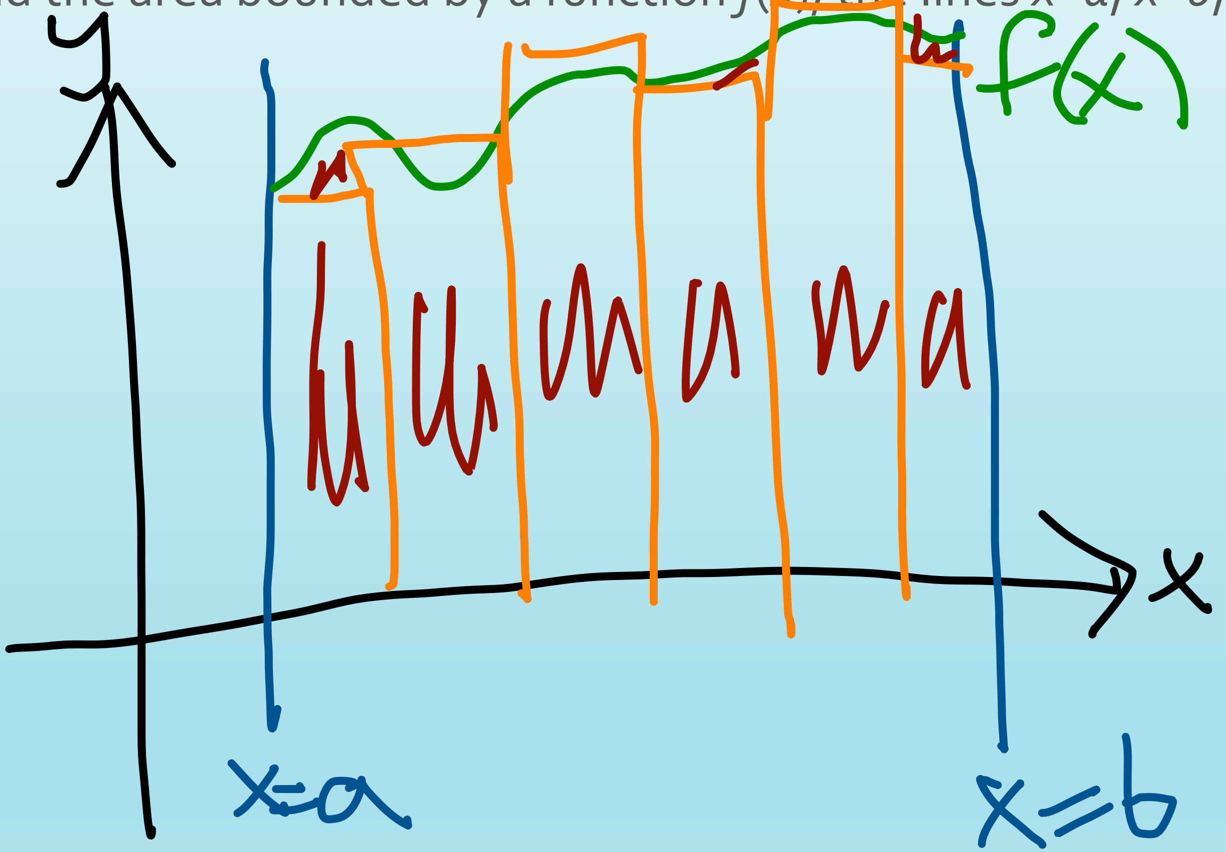
# Learning Goals

- Understand how to partition an interval
- Draw a picture to approximate the area under the curve with a given number of rectangles
- Compute the Upper and Lower sums
- Calculate the midpoint estimate
- average value of $f(x)$ on $[a,b]$ (AV)

# Basic Methodology

geometric interpretation:
"area under the curve"

- *Idea*: Find the area bounded by a function $f(x)$, the lines $x=a$, $x=b$, and the $x$-axis.




area under the curve

approximate area

# Riemann Sums

- *Idea*: Find the area bounded by a function $f(x)$, the lines $x=a$, $x=b$, and the $x$-axis.
- *Procedure*: Break the interval $[a,b]$ into $n$ subintervals, and draw a rectangle in each subinterval.
- Summing the areas of the rectangles will approximate the area under the curve.

# Defining Sigma Notation

We denote the next (finite) sum of terms by:

$$\sum_{i=1}^{n} a_i = \underset{i=1}{a_1} + \underset{i=2}{a_2} + \ldots + \underset{i=N}{a_n}$$

(Practice some sums – examples)

Ex1: $\sum_{k=0}^{5} k = 0+1+2+3+4+5$

Ex2: $\sum_{k=1}^{3} \frac{(-1)^{k+1}}{k+1} = \frac{(-1)^2}{1+1} - \frac{1}{1+2} + \frac{1}{1+3}$

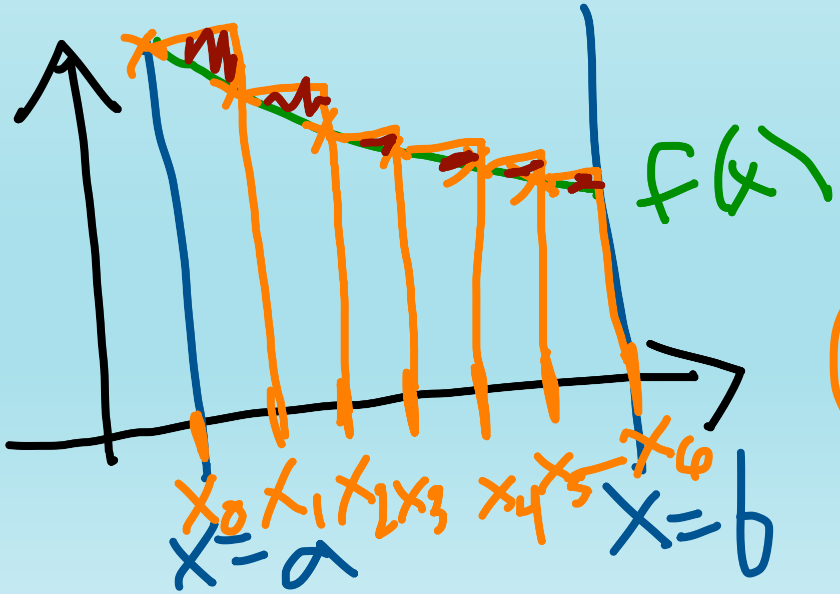
# Riemann Sums (cont.)

Upper estimate: use rectangles that over-approximate the area

Let $M_i = \max\{f(x)\}$ on $[x_{i-1}, x_i]$. (height of rect on $[x_{i-1}, x_i]$)

Then : $U_f = \sum_{i=1}^{n} M_i \Delta x$.

$\left(\Delta x = \frac{b-a}{N}\right)$




(N=6)

approx area

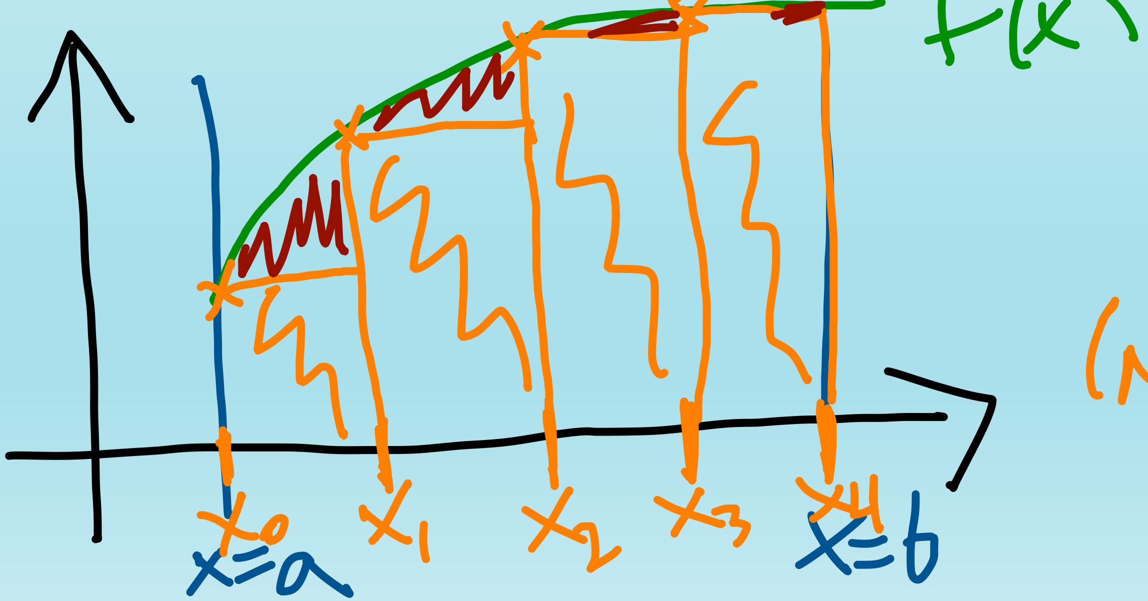
# Riemann Sums (cont.)

Lower estimate: use rectangles that under-approximate the area

Let $m_i = \min\{f(x)\}$ on $[x_{i-1}, x_i]$. (height of the rect on $[x_{i-1}, x_i]$)

Then : $L_f = \sum_{i=1}^{n} m_i \Delta x.$

$\left(\Delta x = \frac{b-a}{N}\right)$




(N=4)

approx area

# Example 1:

We've seen this antiderivative before: $\int_{-1}^{2} f(x)\,dx = \tan^{-1}(x) + \cancel{C}$

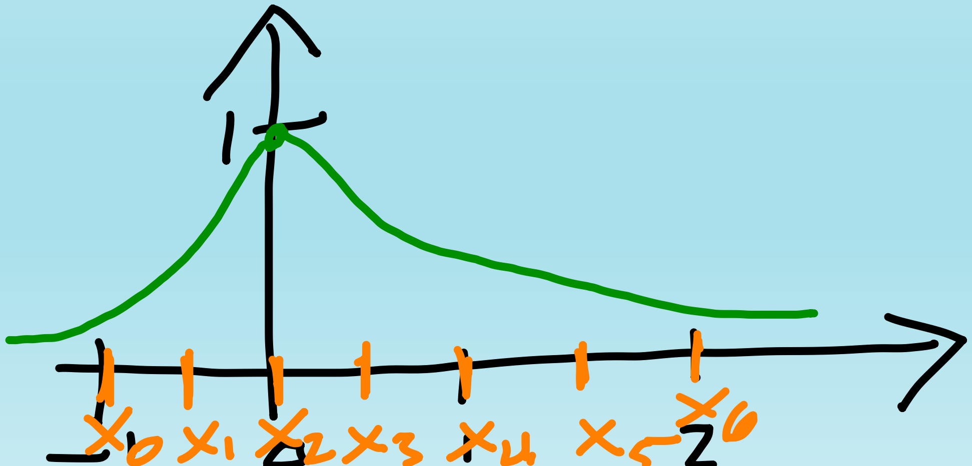
(FTC)

Find the upper and lower sums for the function

(1) $f(x) = \dfrac{(2)\ 1}{x^2+1}$

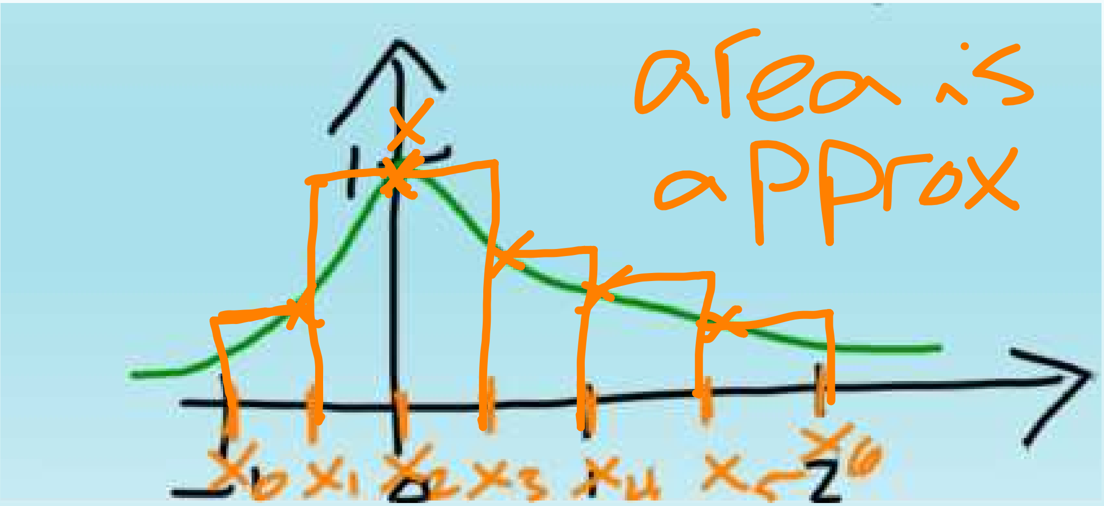
on the interval $[\overset{a}{-1},\overset{b}{2}]$ with $\boxed{n=6}$ subintervals.

$$\Delta x = \frac{b-a}{N} = \frac{3}{6} = \frac{1}{2}$$



## ① upper sum:

$$U_f = \sum_{i=1}^{6} M_i \cdot \Delta x$$




$$= \frac{1}{2}\Big(f(-1/2)+f(0)+f(0)$$

$$+f(1/2)+f(1)$$

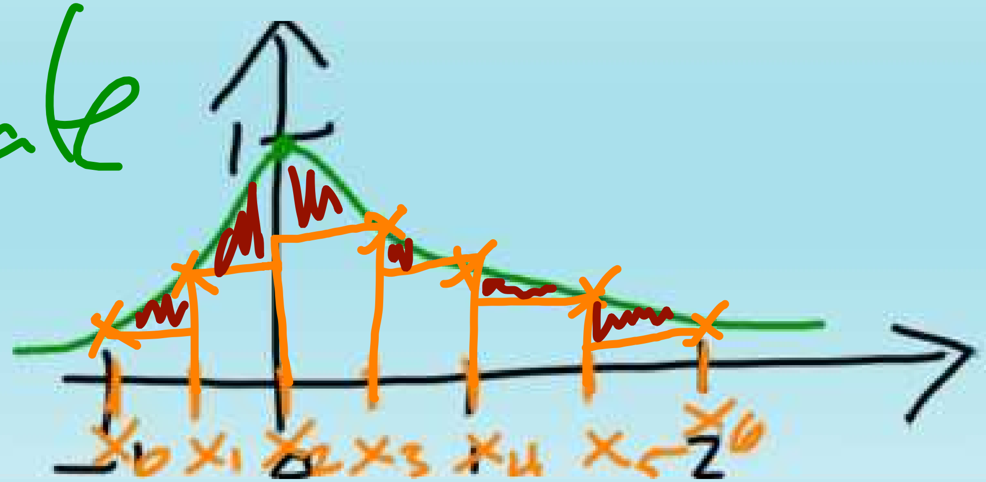
$$+f(3/2)\Big)$$

$$f(x)=\frac{1}{1+x^2}$$

$$U_f = \frac{1}{2}\left(\frac{1}{1+1/4} + 1 + 1 + \frac{1}{1+1/4} + \frac{1}{1+1} + \frac{1}{1+9/4}\right) \rightarrow \text{Plug into calculator}$$

☺

② lower estimate ($L_f$)



$$L_f = \sum_{i=1}^{6} m_i \cdot \Delta x$$

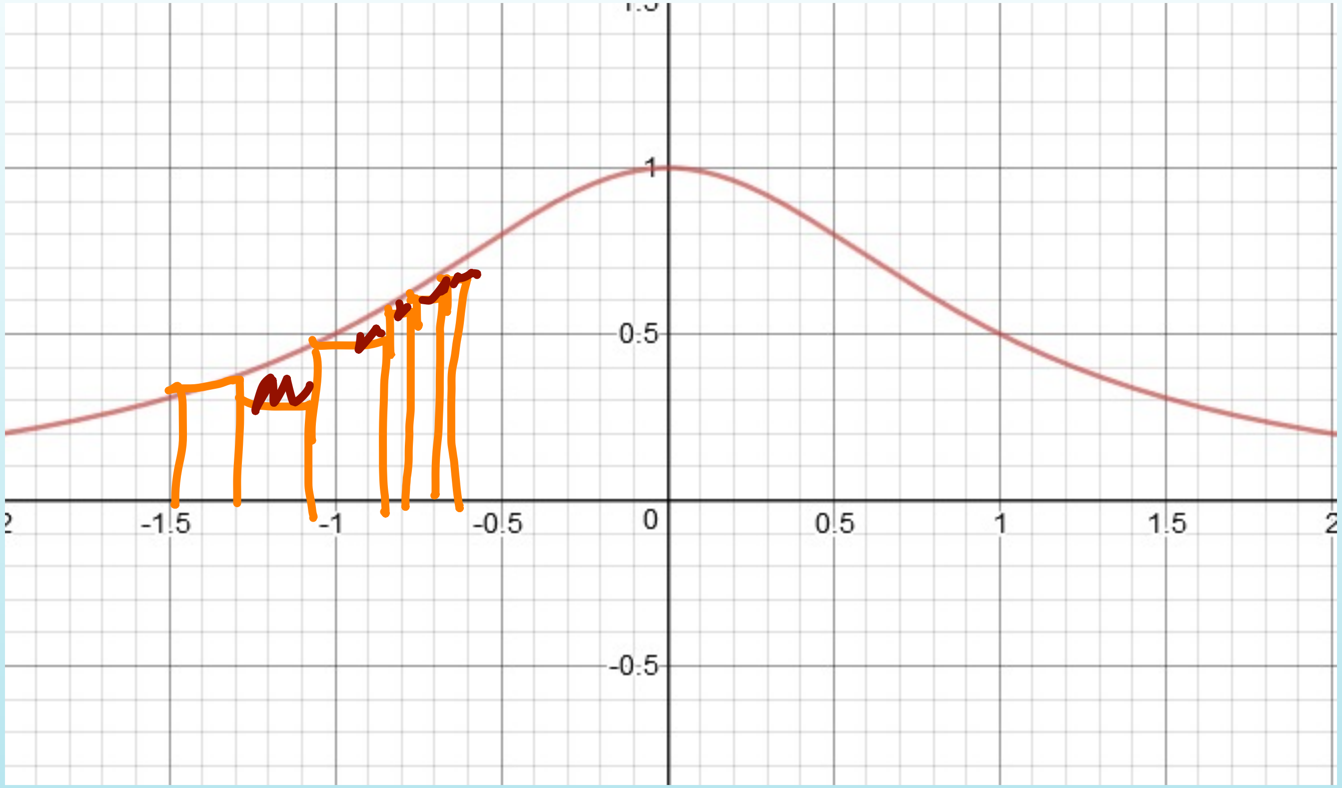
→ x on a graph

$$= \frac{1}{2}\Big( \overset{i=1}{f(-1)} + \overset{2}{f(-1/2)} + \overset{3}{f(1/2)} + \overset{4}{f(1)} + \overset{5}{f(3/2)} + \overset{6}{f(2)} \Big)$$

Pull out the common $\Delta x$

→ <plug-and-chug> :)

$$f(x) = \frac{1}{x^2+1}$$




As we take rectangles of smaller and smaller width, and then add in more of them to fill up the interval evenly, we get close to the area under the smooth curve.

**Key idea:** We will compute a definite integral by writing down a Riemann sum for the area approximation and then take a limit as the size width of the rectangles tends to zero.

$\int_a^b f(x)\,dx$ "definite integral"

# Midpoint Estimate

**Idea:** Go for the middle ground approximation (value in between). Plug in the midpoint of each subinterval.

On the subinterval $[x_{i-1}, x_i]$,

the midpoint is : $\dfrac{x_{i-1} + x_i}{2}$

and the midpoint sum is :

$$M_f = \sum_{i=1}^{n} f\left(\frac{x_{i-1} + x_i}{2}\right)\Delta x$$

# Example 2:

## Find a midpoint estimate to the area from Example 1.

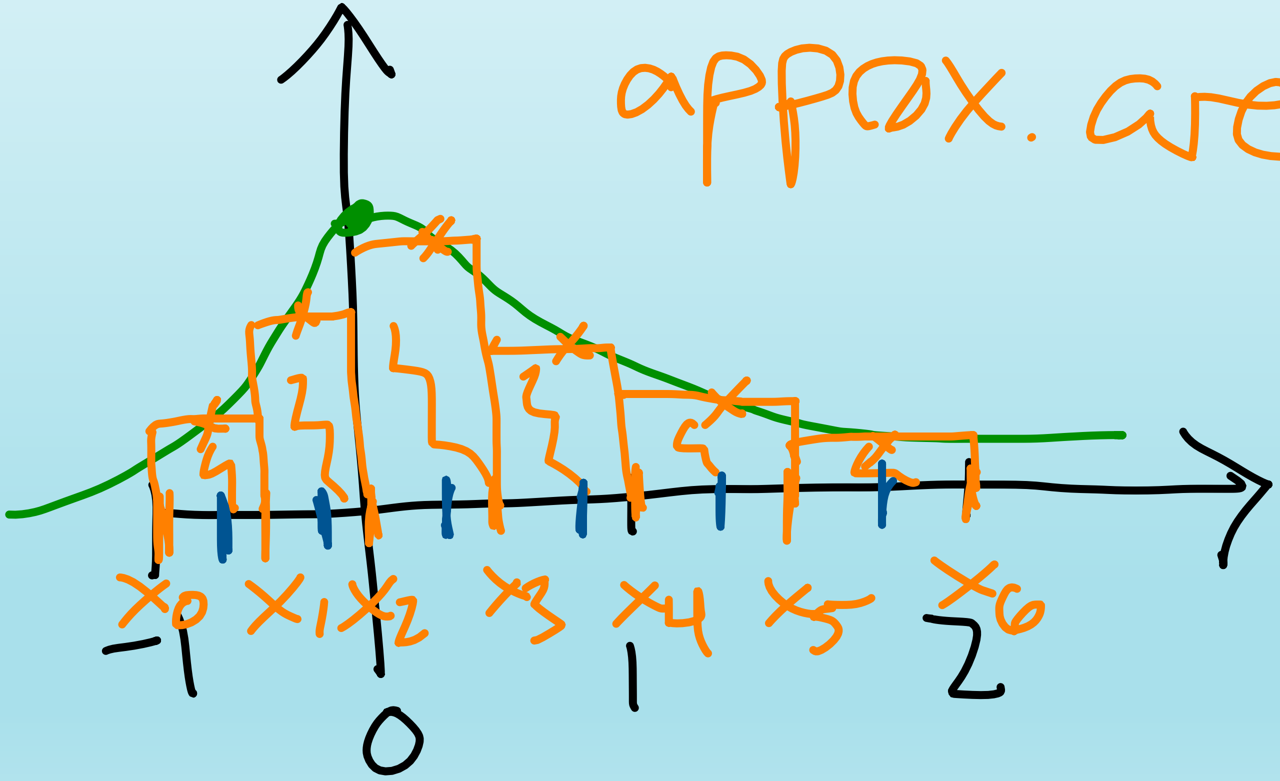
Recall: We want to approximate the area underneath the function

$$f(x) = \frac{1}{x^2+1}$$

on the interval [-1,2] with n=6 subintervals.

$$\Delta x = \frac{b-a}{n} = \frac{1}{2}$$

approx. area



midpoints $\left(\frac{x_{i-1}+x_i}{2}\text{ for each }[x_{i-1}, x_i]\right)$

$\rightarrow\ i = 1, 2, 3, 4, 5, 6$

$i=1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6$

$-3/4,\ -1/4,\ 1/4,\ 3/4,\ 5/4,\ 7/4$

$$M_f = \frac{1}{2}\Big(f(-3/4) + f(-1/4) + f(1/4) + f(3/4) + f(5/4) + f(7/4)\Big)$$

$$\to \langle \text{plug-and-chug} \rangle$$